

Vortex shedding from slender cones at low Reynolds numbers

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Water-tunnel experiments on two cones of taper ratio 36:1 and 18:1 have shown that the frequency of vortex shedding is controlled by the local diameter and has a value which is slightly lower than that on a parallel cylinder of the same size. The shedding process is modulated by a low-frequency oscillation which has a frequency dependent only on U^2/ν and is independent of any physical dimension of the model. A mathematical analogue of the wake, which helps to explain these features of the flow, has been constructed. The pattern of vortices in the wake of a cone suggested by this model is in reasonable agreement with the photographic evidence.

1. Introduction

The present paper gives a brief account of a water-tunnel investigation into the low Reynolds number flow past slender cones with axes normal to the flow. The classical phenomenon of vortex shedding from circular cylinders has been studied in some detail for a considerable period, the various flow régimes have been classified and some explanations for the mode changes have been suggested (see Morkovin 1964, p. 102). However, very little attention has been paid to the influence of a slight three-dimensionality in the model, e.g. taper. The flows past conical models of taper ratio 36:1 and 18:1 were therefore studied. The Reynolds number range of the experiments was 150–600 based on root diameter, or 50–200 based on local diameter in the region where most of the measurements were made.

Roshko (1954) carried out an extensive low Reynolds number investigation of vortex shedding from parallel cylinders. He found that very regular shedding took place over a Reynolds number range of about 47–170; above this upper limit the process became irregular and the frequency difficult to define. He used hot-wire probes in the wake to obtain shedding frequencies within the regular range and found that these followed the empirical law

$$\frac{nd^3}{\nu} = 0.212 \frac{Ud}{\nu} - 4.5, \quad (1)$$

where n is the vortex frequency, d is the cylinder diameter, U is the free stream velocity, ν is the kinematic viscosity. Frequencies given by (1) agree well with early results obtained by Kovasznay (1949).

By comparing the signals from two hot wires spaced across the span Roshko showed that above a certain critical Reynolds number vortices were not shed at the same instant along the whole length of the model. The variations in phase of these signals suggested that the vortices were inclined to the model axis. Berger (1964) also observed that vortices peel off cylindrical models in this way.

Models which are almost parallel and have only slight taper might be expected to generate local flow structures very similar to those on a cylinder of the same diameter. For steady flows this is certainly a good approximation for any central region away from the tip or root. However, if the resulting motion is unsteady, as in the present example of vortex shedding, it is clear that each section of the flow cannot generate an *identical* flow pattern to that of the equivalent parallel model. There is always some coupling between flows at neighbouring stations and if the characteristic time scales differ, the coupling process, and subsequently the motion, will also be affected. At low Reynolds numbers the shed vortices are coherent over many cylinder diameters and the spanwise coupling is strong. Moreover the shedding motion expected from the quasi-cylindrical approach is very sharply defined by a near line spectrum with a centre frequency that varies continuously along the cone. Any spanwise coupling implies that the spectra from neighbouring stations have some common region of overlap and one can therefore expect broadening in the physical situation. The purpose of this experimental study was to attempt to understand this feature of the flow: how can the vortex pattern in the wake of a slender cone be consistent to some extent with the quasi-cylindrical approximation without violating Helmholtz's theorem concerning the continuity of vortex filaments?

2. Apparatus

The experiments were carried out in the National Physical Laboratory closed-circuit water tunnel, which has a vertical working section 13×9 in. (325×225 mm). The flow velocity was controlled by adjusting the fan speed and, at low speeds below about 1 in./sec (25 mm/sec), by restricting the flow with a flap in the return circuit. Within the very low speed range of the present experiments it was necessary to keep the motor rev/min at a minimum and difficulty was experienced in maintaining a constant speed in the tunnel for more than a few minutes at a time.

Figure 1 shows the two brass cone models used in the experiments. Each model was fitted with a large circular disk end-plate to try and isolate the flow on the cone from the wall boundary layer of the water tunnel.

Wake oscillations were detected by a hot-wire probe of 0.0005 in. ($12 \mu\text{m}$) diam. nickel wire soft soldered to phosphor-bronze wire prongs spaced 0.1 in. (2.5 mm) apart. The wire was run in the constant current mode at about 0.2 A. The probe was positioned roughly $\frac{1}{4}$ in. (6 mm) behind the model and to one side of the centre line with the wire normal to the model axis. The hot-wire probe could be traversed so that any station along the cone could be selected from outside the tunnel.

A parallel cylinder of $\frac{3}{8}$ in. (2.5 mm) diam. was used in conjunction with a second hot-wire probe to monitor the tunnel velocity through Roshko's empirical

law relating shedding frequency with flow velocity. After suitable amplification the two hot-wire signals energized the galvanometers of an ultraviolet recorder.

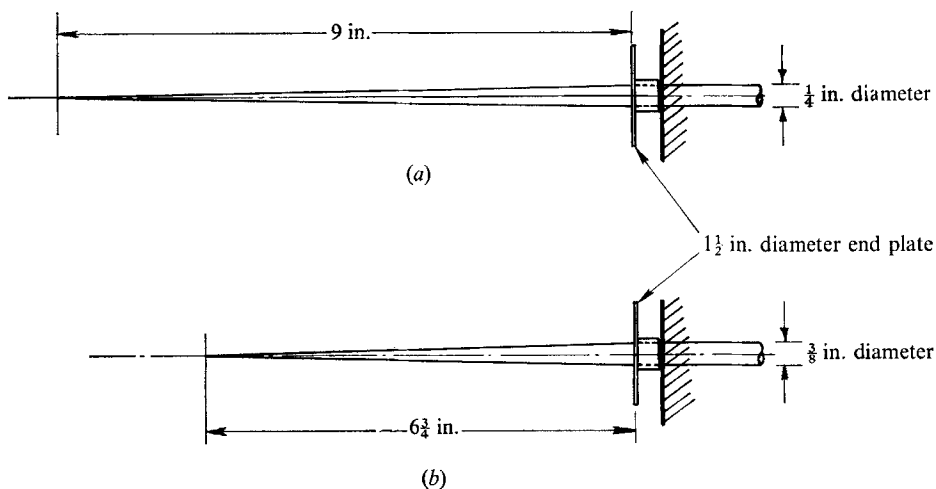


FIGURE 1. Two brass cone models. (a) Model 1, 36 : 1 taper. (b) Model 2, 18 : 1 taper.

3. Experimental results

3.1 Velocity determination

The flow velocities at the monitor station and the model position were compared by replacing the cone with a second parallel cylinder and measuring the shedding frequencies from the two hot-wire records. The difference between the two velocities found with the aid of equation (1) was of the order of 1% and due account of this was taken in reducing the cone data. A typical hot-wire trace from the monitor probe is reproduced in figure 2. Regular constant amplitude signals of this type were obtained up to Reynolds numbers of about 170, above which they became irregular. Tritton (1959), who also used the shedding frequencies of cylinders to find flow velocity, observed changes in the mode of shedding at Reynolds numbers between 80 and 90. As the mode changed the hot-wire signals became irregular and the relationship between flow velocity and shedding frequency also changed slightly. However, in the present experiment no such behaviour was observed and runs with different diameter cylinders over this critical Reynolds number range failed to reveal any derivation from a regular periodic wake. Dye filaments also failed to show the flow patterns obtained by Tritton. The flow visualization did, however, show that above some quite low Reynolds number the vortices were inclined to the model axis, peeling away from the centre as observed by Berger.

3.2 Hot-wire signals from cones

Signals from the hot-wire probe in the wake of the cone models were recorded on ultraviolet-sensitive paper together with those from the cylindrical model. A typical pair of signals is shown in figure 3. The irregular nature of the cone signal

contrasts strongly with the regular periodic motion generated by the cylinder. Clearly the modulation is not associated with fluctuations in velocity as these would also appear on the monitor record. A set of ultraviolet traces from model 1 (18:1) at one spanwise position for a range of flow velocities is reproduced in figure 4. At low Reynolds numbers, when the wake just becomes unstable, the hot-wire signals are fairly regular and periodic. An increase in speed introduces

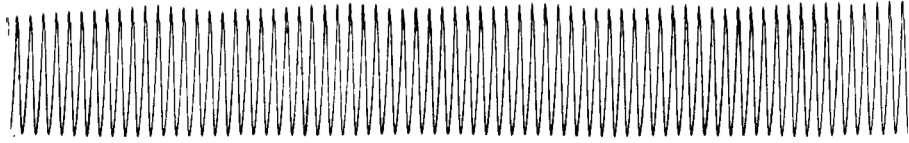
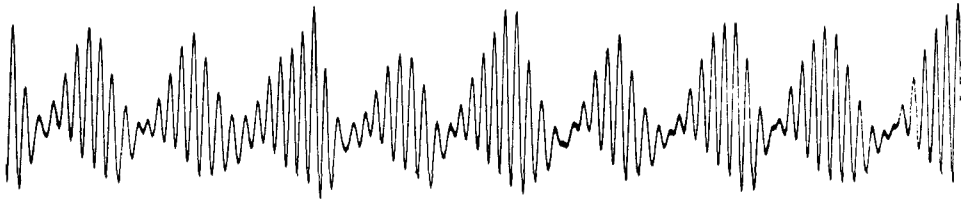
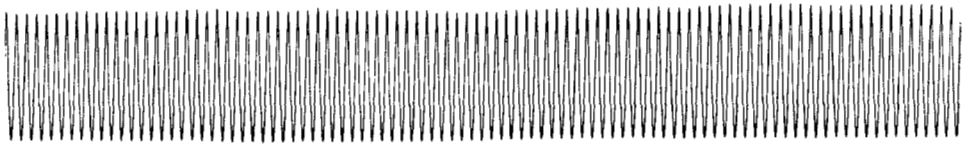


FIGURE 2. Hot-wire signal from the wake region of a circular cylinder in the regular shedding régime.



(a)



(b)

FIGURE 3. Typical hot-wire signals from the wake regions of a cylinder and a cone. (a) From the wake of cone 2. (b) From the parallel model used for determination of velocity.

a low-frequency modulation of the signal, further increases causing the modulation to become deeper until an almost regular series of wave patches is produced. At local Reynolds numbers above about 170, where parallel cylinders start to shed in an irregular manner, the cone signal also becomes less well behaved and the almost periodic modulation gives way to a form which has some added randomness. Shedding frequencies were estimated from regions of the records where the oscillation amplitude was large. These were reduced to a non-dimensional form using the free-stream velocity calculated from the parallel cylinder signal and the local diameter of the cone. Figure 5 compares Roshko's cylinder data with the present measurements from both cones for various spanwise positions. Although this method of obtaining the shedding frequencies is not very precise the results lie within sufficiently narrow limits to show quite clearly that the frequency at any station is solely a function of the local diameter and is

independent of the position along the model. The actual values of the cone shedding frequencies are slightly lower than those from cylinders.

Some of the hot-wire signals were recorded on magnetic tape for subsequent digital processing to produce narrow-band power spectra. The frequencies of the power peaks of these spectra were non-dimensionalized as before and are plotted

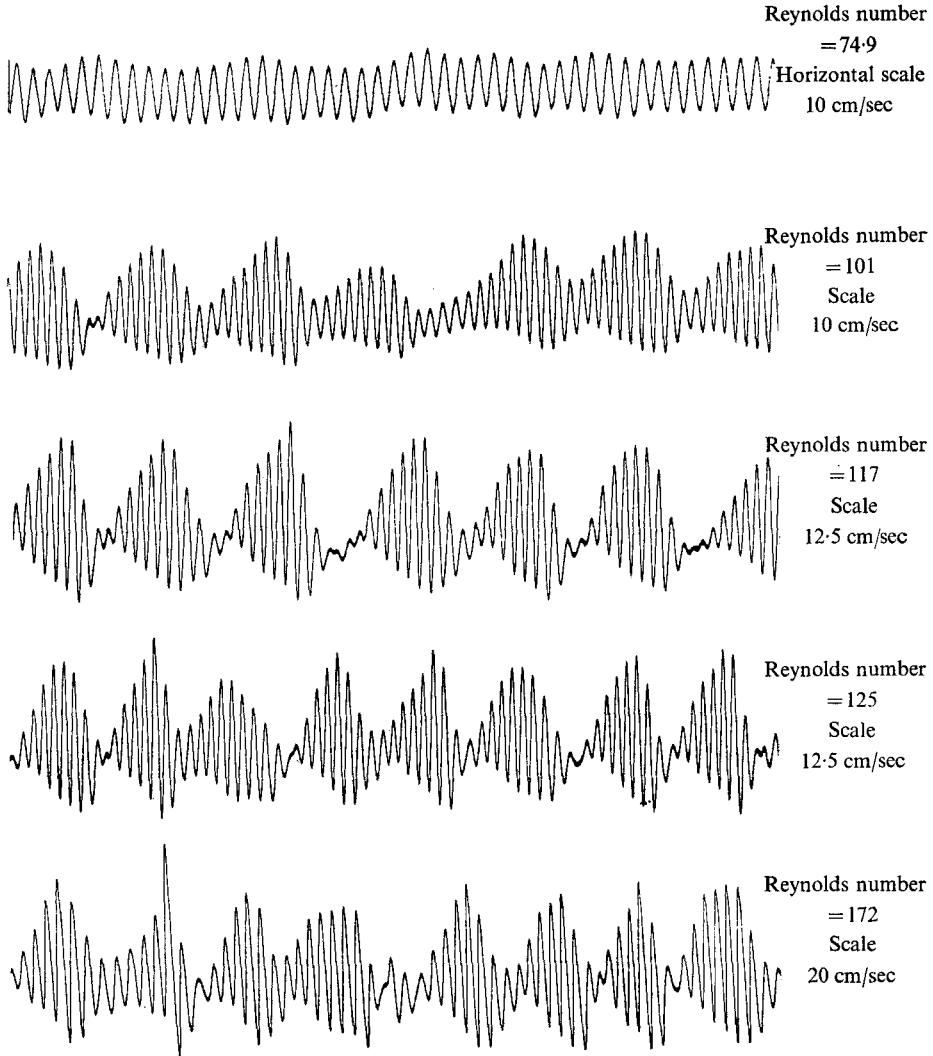


FIGURE 4. Wake oscillations behind cone 2 (18 : 1) for various Reynolds numbers at 5 in. from the base. Local diameter is 0.099 in.

on figure 5 together with data obtained by the simple technique of direct trace measurement.

Since the shedding frequency is controlled by the local diameter on the cone it might be expected that the whole vortex formation process follows the same pattern of similarity. However, signals recorded at different spanwise stations having the same local Reynolds numbers are different in structure and it is clear

that this aspect of the motion, associated with the low-frequency modulation, is not a local phenomenon. The set of signals reproduced in figure 6 were recorded at a fixed flow velocity for various probe positions along the cone. It may be noted how the low-frequency modulation of the shedding signal is maintained at almost the same value throughout the length of the model. Recordings were made with the probe in a fixed position over a range of flow velocities and these are shown on figure 7 in a non-dimensional form based on the free-stream velocity and the cone length. Since nl/U is proportional to Ul/ν , the quantity $n\nu/U^2$ is a constant and the frequency is independent of any physical length scale of the model over the range of parameters investigated.

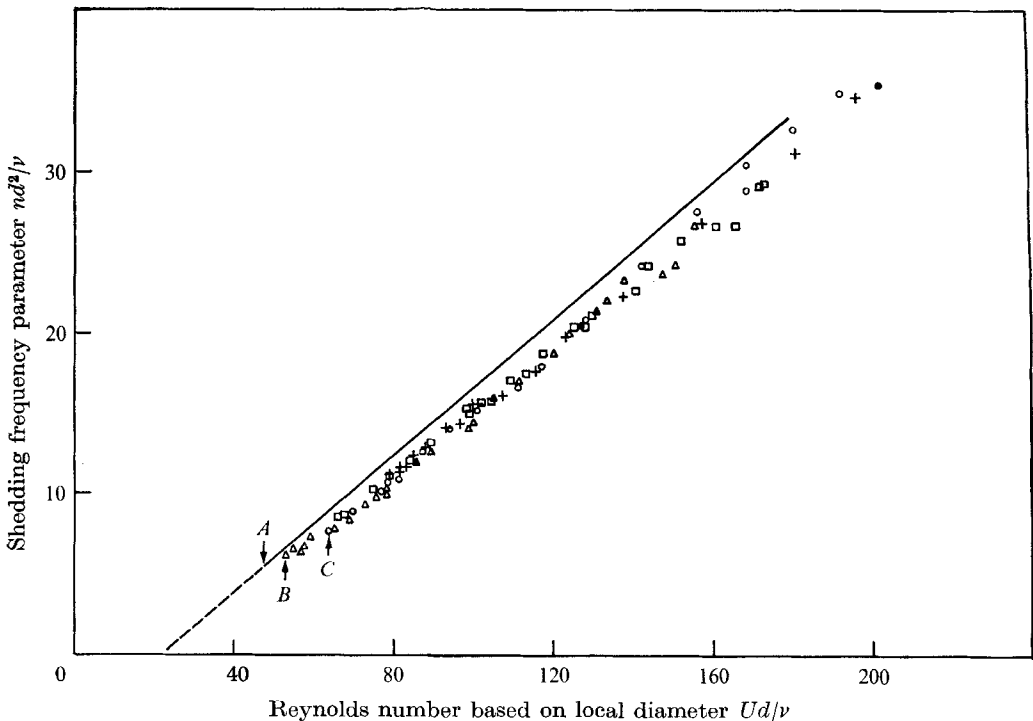


FIGURE 5. Variation of shedding frequency with local Reynolds number. From trace measurement: Δ , 5½ in. from base cone 1; +, 4 in. from base cone 2; \circ , 4½ in. from base cone 2; \square , 5 in. from base cone 2. From power spectra: \bullet , 5 in. from base cone 1; \blacktriangle , 4 in. from base cone 2. —, Roshko's line from cylinders. Point: A, critical Reynolds number for cylinders; B, critical Reynolds number for cone 2 (36 : 1); C, critical Reynolds number for cone 1 (18 : 1).

The tip region of the cone always had a stable laminar wake, but some distance towards the base where the local Reynolds number exceeded some critical value, weak oscillations were observed on the hot-wire signals. This critical local Reynolds number appeared to be higher on the conical models than on parallel ones. Approximate critical local Reynolds numbers are 47 for a cylinder, 53 for the 36 : 1 cone and 65 for the steep 18 : 1 model.

3.3. Flow visualization

Flow visualization with a filament of Fluorescence dye was used to aid the interpretation of the hot-wire measurements. Figure 8 (plate 1) compares the wake structures of a cylinder and cone 2 viewed along the model axes. The modulation

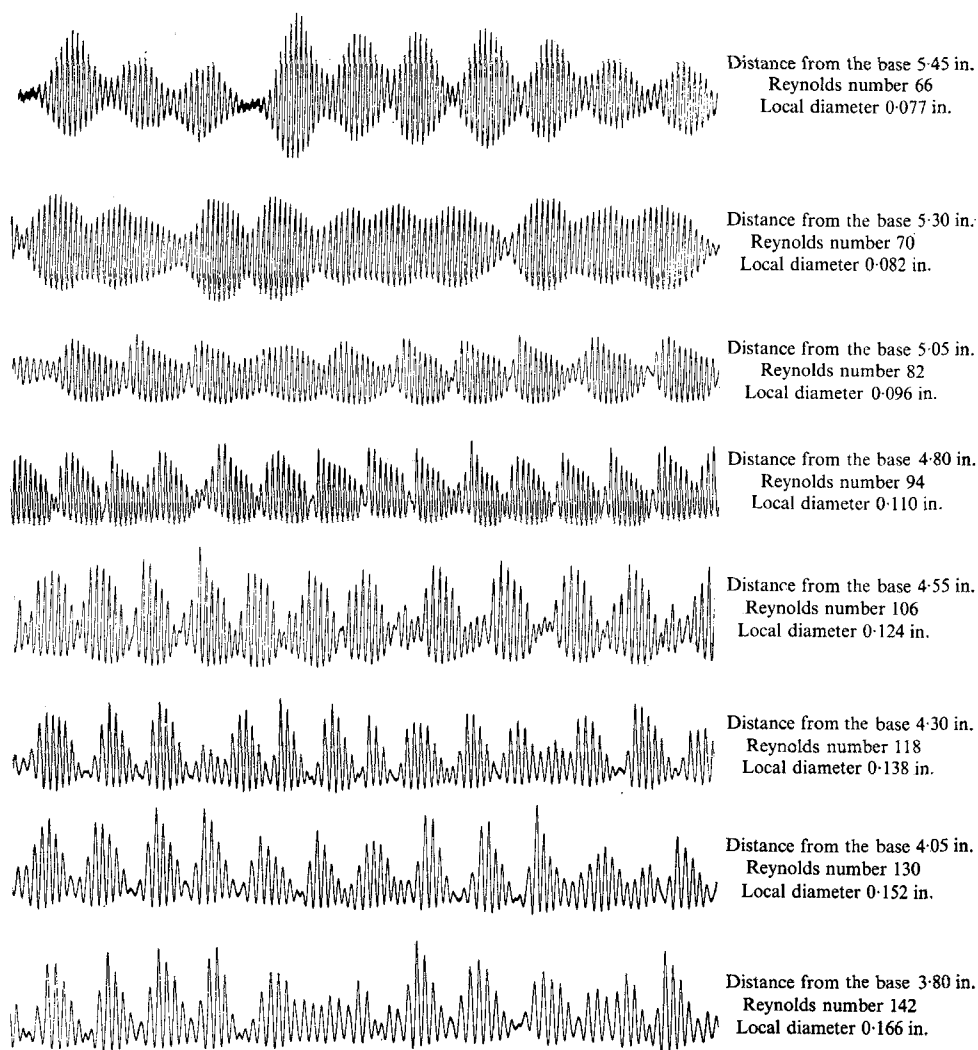


FIGURE 6. Hot-wire recordings taken at a fixed free-stream velocity for a range of span-wise positions along cone 2.

of the regular wake structure by taper is evident, both the wake width and the vortex strength varying down the wake. These distortions form in the base region of the model and then travel downstream with the flow. The almost frozen vortex pattern behind the cylinder persists far downstream, but the cone wake breaks down into a turbulent condition quite rapidly. The dye filaments are not as sharp in the cone photograph as the cylinder, due to the three-dimensional nature of

this flow which causes the dye to spread out along the span of the model. In fact to obtain this photograph it was necessary to illuminate a narrow slice of the wake only. Figure 9 (plate 2), which shows a side view of the flow, gives some indication of the three-dimensional character of the wake. In the region in line with the dye filaments the flow is just unstable and, as there is strong cross-flow in the near wake at these Reynolds numbers, the flow is made visible over quite an extensive portion of the model. Vortices can clearly be seen peeling off the cone with their axes inclined to that of the model. Downstream the pattern becomes confused and the motion appears to break down into turbulence.

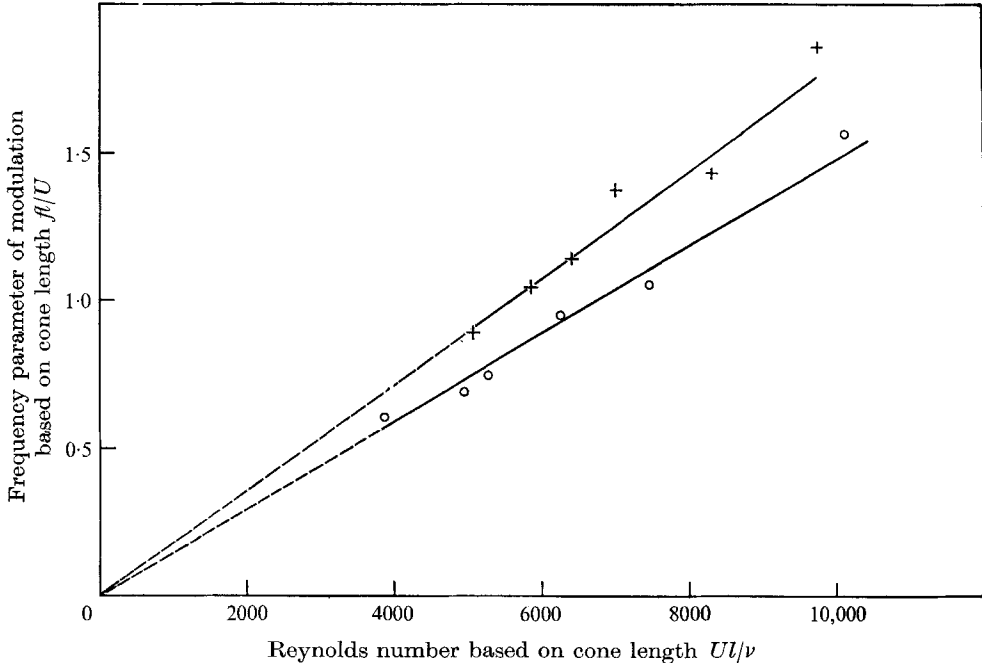


FIGURE 7. Modulation frequency parameter against cone Reynolds number.
+, cone 1; O, cone 2.

4. Theoretical model

Although the complex wake structure of a cylinder is not amenable to a full mathematical treatment, an attempt will be made to discuss the shedding process from a slender cone with the aid of a simple idealized analogue model. In the Reynolds number range 47–170, coherent vortices are shed from parallel cylinders in a periodic manner—the motion is both self-excited and self-limited in amplitude. A Van de Pol oscillator (see Stoker 1950) with a weak non-linear excitation term has just such a behaviour and may represent the motion from a cylinder. The oscillations from a cone can therefore be represented by a line of such oscillators having suitable coefficients to give different oscillation frequencies along the line. It is convenient to neglect the fact that the ‘tuned’ frequency of each section of a cone depends on the local Reynolds number, and only to consider a

simple analogue which has frequency inversely proportional to a local dimension. Such an oscillator can be written

$$\frac{\partial^2\theta}{\partial t^2} D^2 + N^2\theta = \epsilon D \frac{\partial\theta}{\partial t} (\theta^2 - 1), \tag{2}$$

where D is a local length scale, N is the characteristic frequency, and ϵ is a small non-linear parameter determining the initial rate of growth of the motion θ , which represents some velocity fluctuation near the body. To complete the analogue it is necessary to incorporate some coupling between the sections of different natural frequency and this can conveniently be done by including a spanwise stiffness term suitably scaled with a local dimension

$$D^2 \left[\frac{\partial^2\theta}{\partial t^2} - A \frac{\partial^2\theta}{\partial x^2} \right] + N^2\theta = \epsilon D \frac{\partial\theta}{\partial t} (\theta^2 - 1); \tag{3}$$

A is the stiffness parameter.

Putting $D = \alpha x$, where x is the distance from the apex and α is the apex angle, we get the form

$$\alpha^2 x^2 \left[\frac{\partial^2\theta}{\partial t^2} - A \frac{\partial^2\theta}{\partial x^2} \right] + N^2\theta = \epsilon \alpha x \frac{\partial\theta}{\partial t} (\theta^2 - 1). \tag{4}$$

If we consider the problem in the semi-infinite domain ($0 < x < \infty$) we can reduce the above equation to an ordinary differential equation in t/x .

Taking $\eta = \alpha t/x$, we get

$$\theta''(1 - A\alpha^2\eta^2) - 2A\alpha^2\eta\theta' + N^2\theta = \epsilon\theta'(\theta^2 - 1), \tag{5}$$

where primes denote differentiation with respect to η .

The rough form of the solution of the above equation can be deduced sufficiently well for our purpose without the need to solve it exactly. For very slender models (small α) the equation approaches a simple Van der Pol form with the addition of a small damping term proportional to η . When η is small this term is unimportant and the motion builds up into a limited oscillation controlled by the non-linear damping term on the right-hand side, the amplitude stabilizing at such a level that the mean value of the non-linear term is zero. For larger values of η the damping term becomes important and the motion decays. Note that the 'damping' arises solely through the spanwise coupling; there are no dissipative mechanisms involved and the decay of energy at one station occurs through transfer of energy towards the base. Figure 10 shows the form of the solution of equation (5), the rays indicating peaks of the wave-form in the x, t plane. The behaviour of this analogue suggests a possible form of the vortex wake pattern on a semi-infinite cone. The motion shown in figure 10 represents the vorticity shed into the near wake region of the body. This vorticity is convected downstream at some velocity close to that of the free stream and the vortex lines in the physical plane appropriate to this solution are as shown in figure 11.

This analogue of the physical situation can only apply in the Reynolds number range over which the shed vortices are coherent; that is, over some finite length of cone. In fact vortices have a tendency to propagate towards regions of higher

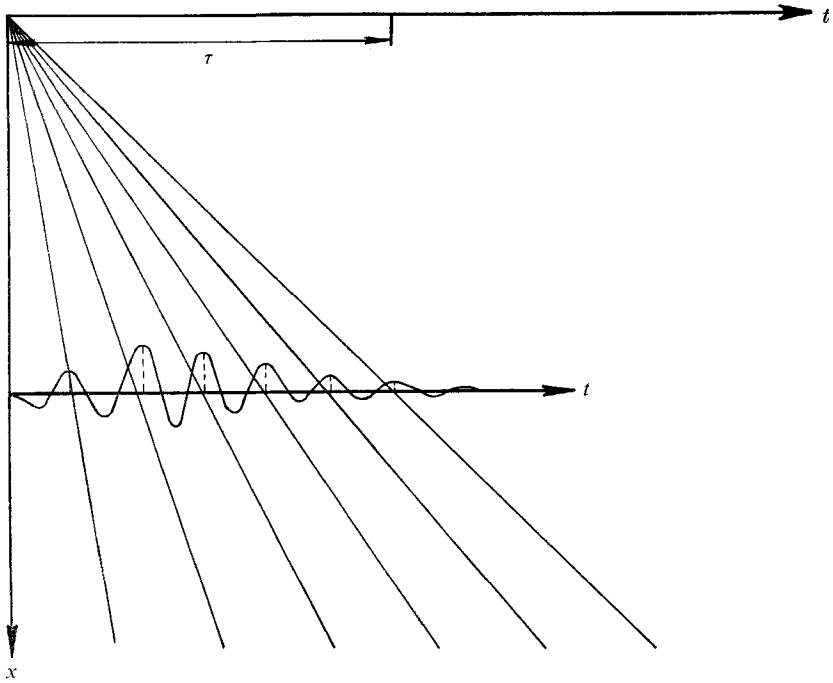


FIGURE 10. Form of solution of equation (8) in the x, t plane. The rays indicate maxima.

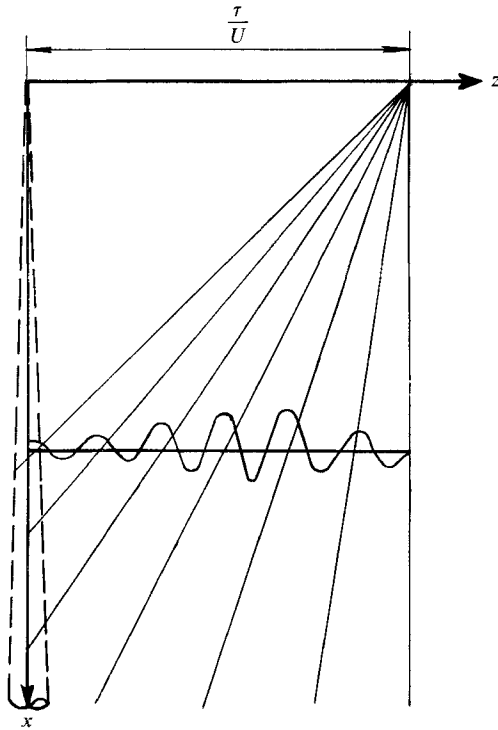


FIGURE 11. Wake structure: after time τ for the unbounded case.

local Reynolds number where turbulence destroys their coherence, preventing them from having any further inhibiting influence on the motion nearer to the tip. It is suggested that the formation, growth and subsequent decay of shed vortex strength follows the behaviour of the semi-infinite system over some finite region of the cone, but after the amplitude of the motion has fallen sufficiently the whole cycle of events is repeated. The period of this secondary oscillation is given by the time taken for the amplitude of the high-frequency motion to fall

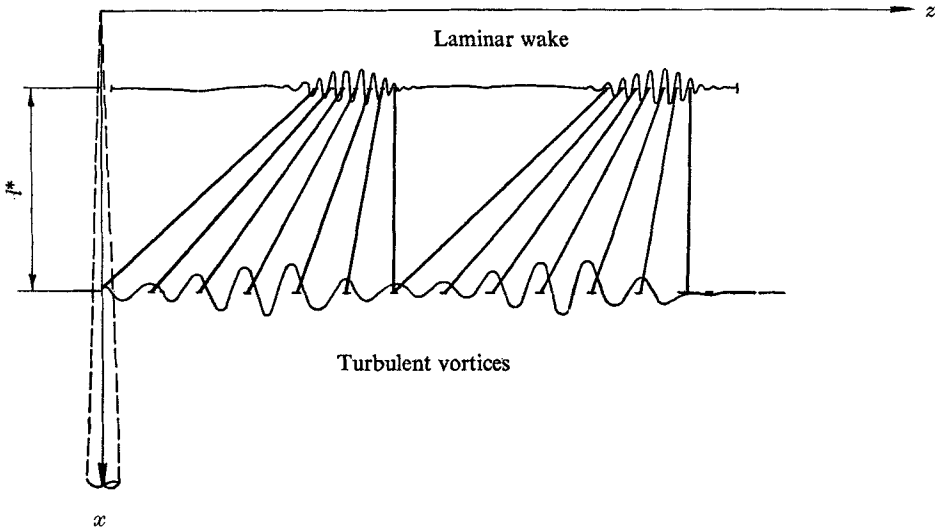


FIGURE 12. Suggested form of wake structure on the physical model.

below some critical level, or for the weakest vortices to be convected into a region of high Reynolds number and hence vortex destruction. Provided the physical length of the cone is sufficient, such a period is proportional to l^*/U , where l^* is the length of the cone over which the vortices are coherent and U is the free-stream velocity. Now, l^* may be defined in terms of a constant Reynolds number Ul^*/ν , so that the repetition frequency becomes proportional to U^2/ν and is independent of the model dimensions. This result is consistent with the experimental observations plotted in figure 7, which also show that the modulation frequency is proportional to U^2/ν .

5. Conclusions

Slight taper on a model of circular cross-section has a powerful influence on the flow structure. Although, as might be expected, the local shedding frequency is determined by the local diameter, coupling between regions of different characteristic frequency introduces a certain amount of amplitude modulation to the motion. Modulation of the shedding process is inevitable if coherent vortices are shed at different rates along the cone. The present series of experiments have shown that the modulation frequency is independent of the spanwise station.

The vortices are shed in patches with the predominant frequency in the patch varying continuously along the model while the repetition frequency of the packets remains constant.

An attempt has been made to provide a mathematical representation of the flow by considering a continuous line of non-linear oscillators with different tuned frequencies which are coupled together. In the unbounded case this model shows how spanwise coupling can introduce a damping effect on the oscillation at any one station. It is postulated that the physically finite situation can be represented by a series of signal packets described by solutions for the semi-infinite case. Although no solutions of the equations have been obtained this approach enables a physical picture of the vortex wake pattern to be constructed. This picture is consistent both with the hot-wire measurements and with the flow visualization observations.

A slight taper, or possibly shear in the oncoming stream, can produce an irregular modulation of the shedding signal not unlike those shown by Tritton (1959). His flow visualization photographs are also fairly similar to some of those obtained on a cone and it is tentatively suggested that the observed breakdown in regular shedding which he observed at Reynolds numbers of between 80 and 90 may have been due to some non-uniformity in the flow.

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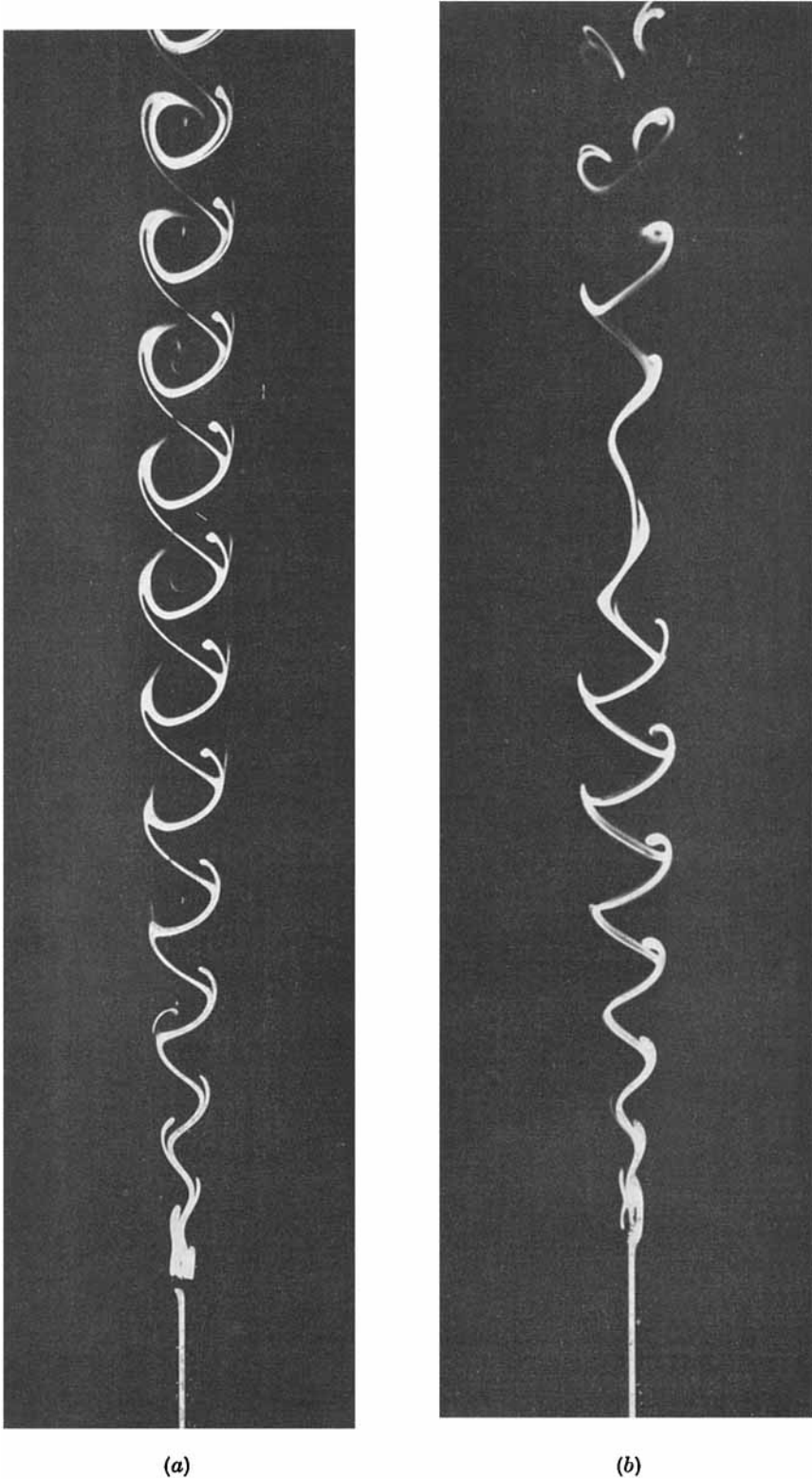


FIGURE 8. Dye visualization of the wake structures behind (a) a circular cylinder and (b) cone 2 (18 : 1) viewed along the model axes.

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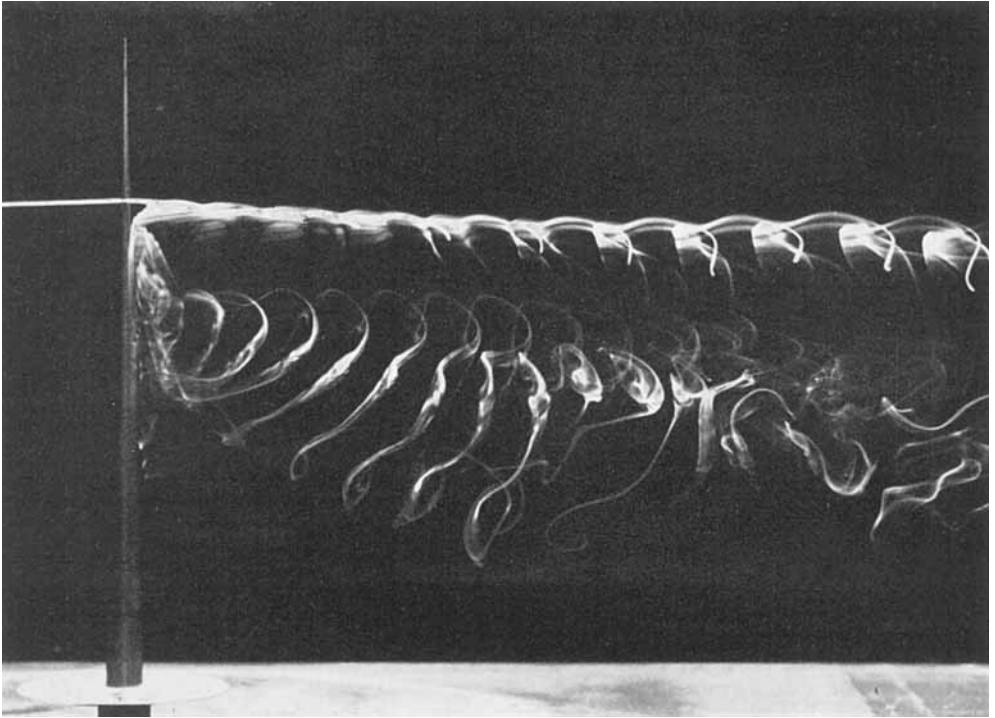


FIGURE 9. Side view of wake behind cone 2.