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A THEORETICAL APPROACH FOR ANALYZING THE RESTABILIZATION OF WAKES

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Abstract

A linear stability analysis provides the basis for an analysis of the effects of prescribed changes upon the global stability characteristics of a bluff body wake. The analysis is applied in particular to the problem of placing a small cylinder in the wake of a larger one in order to suppress the vortex street. A prediction of the alteration in the critical Reynolds number of the system as a function of the location of the second cylinder relative to the first is presented and is shown to compare favorably with experimental results[1]. The effect upon shedding frequency is also considered. The approach relies on the solution of an adjoint eigenvalue problem and an inhomogeneous adjoint boundary value problem. These adjoint solutions are shown to be a key component in the analysis of a variety of wake control strategies.

Introduction

The global dynamics of a wake, such as that behind a circular cylinder, can be changed dramatically with small changes in flow conditions. This is seen clearly in the striking low- Reynolds number experiments of Strykowski and Sreenivasan[1] (here onwards referred to as SS) where the periodic vortex shedding in the wake of a cylinder is suppressed by placement of a second smaller cylinder at appropriate locations.

Recognizing the potential of utilizing this susceptibility to small changes as a means of achieving flow control, researchers have introduced two principal strategies. The simplest strategy for encouraging the restabilization of a flow involves the introduction of some fixed change, such as steady wall suction for a boundary layer flow. A second more sophisticated approach involves adaptive control whereby a feedback controller measures some physical parameter of the system, processes that information, and then generates an appropriate unsteady excitation in response. This has been used successfully by Liepmann et al.[2] to suppress Tollmien-Schlichting wave packets in a boundary layer by active wall heating.

The question of how receptive an instability is to control is an additional consideration, especially in the highly non-parallel near wake of a cylinder or other bluff body. The same control strategy implemented at one location in the flow may have quite a different effect at another. This is a major new feature which was not a serious concern in the study of parallel flows.

Steady and unsteady feedback control, and variations in receptivity to control, are all significant influences in the dynamics of the circular cylinder wake when a second small cylinder is immersed. As a result it is an excellent problem to study in order to understand the role of these different issues.

We present here a general linear analysis of the effect upon global stability resulting from the introduction of steady and unsteady feedback control forces upon the wake of a circular cylinder. Experimental results of SS support strongly the contention that a global temporal-mode approach provides an appropriate description of the initial growth stage of vortex shedding. Our particular interest, following SS, is in the restabilizing effect of placing a second smaller cylinder in the wake of the first, where in fact both steady and unsteady control forces are in play.

Implicit in the current analysis is the fact that the local far-wake profile is 'convectively unstable' and is driven by the 'globally-unstable' near wake (see Tryantafyllou et al.[3], and Yang et al.[4], and Monkewitz et al.[5]). As a consequence the immediate wake of the body is where the unstable disturbances are most receptive to control; a feature which emerges naturally from the analysis and

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is quantified.

We consider the wake of an isolated cylinder to be our basic system, and treat the presence of a second small cylinder in the flow as a 'feedback controller', with both a steady and unsteady control contribution. The mean drag on the small cylinder is associated with a steady reaction force upon the fluid. Likewise, unsteady motion past the small cylinder induces an unsteady reaction. The current analysis is used to determine the change in the global stability characteristics, i.e., changes in the eigenvalues and the critical Reynolds number for the flow, once an appropriate model for the feedback forces due to the small cylinder has been defined.

A standard linear stability calculation for viscous flow past a circular cylinder is the first step in the analysis. The essential wake dynamics is captured using a single mode with the computational domain chosen sufficiently large. In order to calculate the changes in stability as a result of control a pair of adjoint problems must be solved. In order to handle unsteady control forces we solve an adjoint eigenvalue problem, whose formulation is familiar to researchers in bifurcation theory (see for example Iooss and Joseph[6]) though the employment within this context is believed to be new. In addition an inhomogeneous adjoint boundary value problem is formulated, using principles from the calculus of variations, and whose solution is used to model the effect of steady control. A description of these problems, and how their solutions may be utilized, forms the main body of the analysis presented here.

This approach is áttractive since there is no need to calculate explicitly the correction to the flow field due to the presence of the small cylinder at some location; with a direct calculation the correction to the global stability can be found for the small cylinder at *any* location. A further bonus is the fact that the same adjoint solutions can be employed in the analysis of the effect of a variety of control strategies.

Linear stability and adjoint eigensolutions

The steady two-dimensional incompressible viscous flow field past a circular cylinder is taken as the basic flow for our problem. It is obtained by solving the full Navier-Stokes equations using a stream function formulation following the spectral approach of Zebib[7]. The problem is solved on an annular domain centered on the cylinder with soft outflow boundary conditions[7] applied at 20 cylinder radii. Here 44 Chebychev polynomials are used in the radial direction, and 40 Fourier components in the circumferential direction. The velocity field $\underline{V}(\underline{r})$ has been normalized by the uniform flow at infinity $\tilde{V}_\infty$, and pressure $P(\underline{r})$ by $\rho\tilde{V}_\infty^2$ where ρ is the fluid density. The Reynolds number R is based upon cylinder diameter, with lengths scaled on the cylinder radius $d/2$ and time on $d/2\tilde{V}_\infty$. The calculated separation angles and wake lengths agree with the results of Dennis et al.[8].

If linear time-dependent velocity and pressure disturbances of the form $(\underline{v}(\underline{r}), p(\underline{r}))e^{i\omega t}$ (using a complex representation) are superimposed upon this field, then they satisfy the linearised Navier Stokes equations and continuity:

$$i\omega\underline{v} + L(\underline{V}; R)\,\underline{v} + \nabla p = \underline{0}, \tag{1}$$

$$\nabla.\underline{v} = 0, \tag{2}$$

where the ith component of the linear operator $L(\underline{V}; R)$ is

$$(L(\underline{V}; R)\,\underline{v})_i = V_j\frac{\partial v_i}{\partial x_j} + v_j\frac{\partial V_i}{\partial x_j} - \frac{2}{R}\frac{\partial^2 v_i}{\partial x_j^2}. \tag{3}$$

The condition of zero disturbance is applied on the cylinder walls and at the outer perimeter of the computational domain. This defines an eigenvalue problem whose numerical solution yields a discrete set of eigensolutions. It is recognized that the imposition of a zero disturbance condition at the outer perimeter is an arbitrary choice. However, this constraint is not expected to affect the near wake dynamics greatly if the outer perimeter is sufficiently far away, especially in view of the convective behaviour of the far wake.

The stream functions for the eigensolutions (with 34 radial Chebychev polynomials and 30 circumferential Fourier components) are obtained using a modification of the approach of Zebib. For the least stable mode $(\underline{v}^{(0)}, p^{(0)})$ with eigenfrequency ω_0, Figure 1 shows the plot of dimensionless growth rate $-Im(2\omega_0 R)$ versus Reynolds number, with Figure 2 showing the corresponding frequency $Re(\omega_0 R/\pi)$, both being compared to the results of SS. The inability of computations of various kinds to mirror precisely the correct Strouhal frequency behavior is common[9], though it should be remembered that the current analysis is linear and may not describe accurately the nonlinear limit cycle (The Strouhal frequency is computed here to be 0.12). The growth rate correlation is very good, aside from the offset in the location of the critical Reynolds number. The relation

$$Im(\omega_0) = -\frac{0.1}{R}(R - R_c) \tag{4}$$

provides a very good representation of the growth rate/Reynolds number relationship. The experimental results of SS give the critical Reynolds number R_c to be 46, while for the current computation $R_c = 50$.

The key to the analysis of the effects of changes lies in the use of adjoint equations. The idea of the adjoint can be traced back to Lagrange, and in recent times it has found considerable use in bifurcation theory[6]. Ince[10] provides good background reading in this area. The solution to a carefully-formulated adjoint problem can represent a Frechet or functional derivative, which relates how small changes in one flow quantity are related to changes in some other. For the current problem we wish to study how the growth rate associated with the vortex shedding motion varies as a result of feedback control at different locations within the flow. The first of the adjoint problems needed to address this question is the adjoint eigenvalue problem which we formulate here after developing the adjoint linearised Navier-Stokes operator. (A second inhomogeneous adjoint boundary value problem is formulated later in the section on modification of flow stability.)

We begin the construction of the adjoint equations by defining an inner product as follows. For a pair of complex vector fields $\underline{u}$, and $\underline{v}$ defined over the flow domain D, the inner product is defined after Ladyzenskaya[11] to be

$$[\underline{u}, \underline{v}] = \int_D \underline{u}.\underline{v} dS. \tag{5}$$

For any solenoidal fields $(\underline{v}, p)$ and $(\underline{v}^*, p^*)$, defined over the flow domain, we now construct a Lagrange identity

$$\left(L(\underline{V}; R)\, \underline{v} + \nabla p\right). \underline{v}^* - \underline{v}. \left(L^*(\underline{V}; R)\, \underline{v}^* - \nabla p^*\right) = \nabla.\underline{M}(\underline{v}, \underline{v}^*), \tag{6}$$

where $L^*(\underline{V}; R)$ is the adjoint linearised Navier-Stokes operator with components

$$\left(L^*(\underline{V}; R)\, \underline{v}^*\right)_i = -V_j \left\{ \frac{\partial v_i^*}{\partial x_j} + \frac{\partial v_j^*}{\partial x_i} \right\} - \frac{2}{R} \frac{\partial^2 v_i^*}{\partial x_j^2}. \tag{7}$$

The vector $\underline{M}(\underline{v}, \underline{v}^*)$ is the bilinear concomitant with components

$$M_j = -\sigma_{ij} v_i^* + v_i \sigma_{ij}^*, \tag{8}$$

where

$$\sigma_{ij} = -p\delta_{ij} - V_j v_i + \frac{2}{R} \left\{ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right\}, \tag{9}$$

$$\sigma_{ij}^* = p^*\delta_{ij} + V_k v_k^* \delta_{ij} + \frac{2}{R} \left\{ \frac{\partial v_i^*}{\partial x_j} + \frac{\partial v_j^*}{\partial x_i} \right\}. \tag{10}$$

The factor σ_{ij} is the deviatoric stress tensor associated with the linear field $(\underline{v}, p)$. The tensor σ_{ij}^* will be referred to as the adjoint stress tensor.

Consider the eigensolutions of the homogeneous adjoint problem for which

$$i\omega^* \underline{v}^* + L^*(\underline{V}; R)\, \underline{v}^* - \nabla p^* = \underline{0}, \tag{11}$$

$$\nabla.\underline{v}^* = 0, \tag{12}$$

subject to the condition $\underline{v}^* = \underline{0}$ on the cylinder and the outer boundary of the flow domain. By virtue of the fact that our choice of boundary conditions for $\underline{v}$ and $\underline{v}^*$ makes the normal component of the bilinear concomitant, $\underline{M}$, zero everywhere on the flow boundary, the frequency spectrum resulting from the solution of the adjoint problem is identical to that of our original eigenvalue problem. Furthermore, if ω_n is the nth eigenvalue, and $\underline{v}^{(n)}$ and $\underline{v}^{(n)^*}$ the corresponding nth eigen-velocity field and its adjoint, then the orthogonality condition

$$\left[\underline{v}^{(n)}, \underline{v}^{(m)^*}\right] = \delta_{nm} \tag{13}$$

is satisfied. This is a useful standard result from bifurcation theory[6]. We will denote the adjoint eigensolution to the least stable mode by $(\underline{v}^{(0)^*}, p^{(0)^*})$.

Given that the eigensolution $(\underline{v}^{(0)}, p^{(0)})$ with eigenfrequency ω_0 characterises the unsteady and possibly unstable motion of the cylinder wake, we wish to know the effect upon ω_0 of the placement of a second small cylinder in the wake of the first. A direct approach would involve the choice of a position and diameter for the small cylinder, the recomputation of the steady flow field using a nonlinear Navier-Stokes solver, and subsequently a global stability analysis. This is clearly a prohibitively-expensive approach for more than a few cylinder locations and diameters.

The alternative which we pursue here is to treat the small cylinder as a feedback controller, the effect of the control being to modify the global stability characteristics. The change induced by the small cylinder is represented here as a linear perturbation of the eigensolution $(\underline{v}^{(0)}, p^{(0)})$. With a direct calculation the linear change in ω_0 is determined for the small cylinder at any location within the flow domain.

A model for the small cylinder

The small cylinder is modelled here by a point source of momentum, with strength and direction equal and opposite to the force which the small cylinder experiences. A model must be prescribed for this force, and we make an approximation based upon local instantaneous flow conditions. The use of instantaneous flow conditions is justified since the timescale of unsteadiness is based upon the large cylinder Strouhal frequency, with the wake of the small cylinder adjusting typically of the order of the diameter ratio $1/\eta = d/d_s$ faster, d_s being the diameter of the small cylinder. The Reynolds number for the small cylinder is of the order of ηR, which for our calculation is typically in the range 0 to 10. In this range the force experienced by the cylinder is proportional to the velocity. The constant of proportionality is determined here from a semi-empirical rule for the drag coefficient for a cylinder in a low-Reynolds number flow.

If $\tilde{\underline{V}}_s$ is the total instantaneous velocity in the vicinity of a small cylinder of diameter d_s, then the force upon such a cylinder immersed in an infinite uniform steady flow of velocity $\underline{\tilde{V}}_s$, is

$$\frac{1}{2}\rho d_s C_D(R_s^{inst})|\underline{\tilde{V}}_s|\underline{\tilde{V}}_s, \tag{14}$$

where C_D is the drag coefficient, and

$$R_s^{inst} = \frac{|\underline{\tilde{V}}_s|}{\tilde{V}_\infty}\eta R \tag{15}$$

is the instantaneous Reynolds number based upon the small cylinder diameter.

Normalised by an appropriate factor $2/(\rho\tilde{V}_\infty^2 d)$, the force exerted upon the flow by a small cylinder placed at location $\underline{r}_0$ is modelled as a source in the dimensionless momentum equations

$$\underline{F}_s = -\eta C_D(R_s^{inst})|\underline{V}_s|\underline{V}_s\delta(\underline{r}-\underline{r}_0), \tag{16}$$

where $\underline{V}_s = \underline{\tilde{V}}_s/\tilde{V}_\infty$. If we express $\underline{V}_s$ as a sum of steady and unsteady components, $\underline{V}$ and $\underline{v}$ respectively, then we can split $\underline{F}_s$ likewise as

$$\underline{F}_s = \underline{F}\delta(\underline{r}-\underline{r}_0) + \underline{f}\delta(\underline{r}-\underline{r}_0). \tag{17}$$

An expansion on the assumption that $\underline{v}$ is small reveals that

$$\underline{F} = -\eta C_D(R_s)|\underline{V}|\underline{V}, \tag{18}$$

and

$$\underline{f} = -\eta\left\{\eta R\left.\frac{dC_D}{dR}\right|_{R=R_s}(\underline{V}.\underline{v})\underline{V} + C_D(R_s)\left(|\underline{V}|\underline{v} + \frac{(\underline{V}.\underline{v})}{|\underline{V}|}\underline{V}\right)\right\} + O(|\underline{v}|^2), \tag{19}$$

where now $R_s = |\underline{V}|\eta R$ is the local Reynolds number based on the steady velocity component.

The drag coefficient is approximated well in the Reynolds number range 1 to 10 by

$$C_D(R) = \frac{1}{R(a+b\ln R)}, \tag{20}$$

with $a = 0.0798$ and $b = -0.0194$. The form of this relation is taken from asymptotic low Reynolds number analysis for flow past a cylinder (see Rosenhead[12]), with the constant a taking the theoretically-derived value. The constant b can also be derived theoretically, but fails to give satisfactory agreement with experiments at the 'relatively high' Reynolds number of 5! The constant b is chosen to fit the experimental and computational data presented by Dennis et al.[8], and so relation (20) is in essence semi-empirical. Using (20) in (19) we find that

$$\underline{f}(\underline{v}) = -\epsilon(R_s)\underline{v} + O(\epsilon^2|\underline{v}|, \epsilon|\underline{v}|^2), \tag{21}$$

where the coefficient

$$\epsilon(R_s) = \eta C_D\left(|\underline{V}(\underline{r}_0)|\eta R\right)|\underline{V}(\underline{r}_0)| = \frac{1}{R(a+b\ln R_s)}, \tag{22}$$

reflects the linear drag/velocity relationship which is appropriate at the low Reynolds numbers present here. We can also rewrite (18) simply as

$$\underline{F}(\underline{V}) = -\epsilon(R_s)\underline{V}. \tag{23}$$

Modification of wake stability

The presence of the small cylinder will lead to a modification to both the steady and unsteady flow components. If we write $(\underline{V}^{(0)}, P^{(0)})$ for the steady flow past the original cylinder, then the new steady flow is written in terms of an expansion around this flow. The expansion is made on the basis that the maximum value taken by ϵ, which we will denote by $\hat{\epsilon}$, is much less than 1 (which is true for the current problem), with

$$\underline{V} = \underline{V}^{(0)} + \hat{\epsilon}\underline{V}' + O\left(\hat{\epsilon}^2\right), \tag{24}$$

$$P = P^{(0)} + \hat{\epsilon}P' + O\left(\hat{\epsilon}^2\right), \tag{25}$$

for some steady fields $\hat{\epsilon}(\underline{V}', P')$. At order $\hat{\epsilon}$ the linearisation of the steady Navier-Stokes equations gives

$$L\left(\underline{V}^{(0)}; R\right)\hat{\epsilon}\underline{V}' + \nabla\hat{\epsilon}P' = \underline{F}(\underline{V}^{(0)})\delta(\underline{r}-\underline{r}_0), \tag{26}$$

$$\nabla.\left(\hat{\epsilon}\underline{V}'\right) = 0, \tag{27}$$

where $\underline{V}' = \underline{0}$ on the cylinder surface, and ideally should satisfy the soft boundary conditions at the outer flow boundary. Here we impose the hard conditions $\underline{V}' = \underline{0}$. This is assumed not to have a large effect since the domain appears $1/\eta \sim 10$ times larger for the small cylinder than for the large cylinder.

The equations of unsteady motion linearised around this perturbed base flow can now be written with the left hand side consisting of the governing equations for disturbances superimposed on the original flow $\underline{V}^{(0)}$. On the right hand side are placed the terms accounting for the dynamic effect of the steady flow perturbation and the unsteady dynamic forces which will arise due to the small cylinder :

$$i\omega\underline{v} + L\left(\underline{V}^{(0)}; R\right)\underline{v} + \nabla p = \underline{Q}(\underline{v}) + O\left(\hat{\epsilon}^2\right), \tag{28}$$

where

$$Q_i(\underline{v}) = f_i(\underline{v})\delta(\underline{r}-\underline{r}_0) - \left\{\hat{\epsilon}V_j'\frac{\partial v_i}{\partial x_j} + v_j\frac{\partial}{\partial x_j}\left(\hat{\epsilon}V_i'\right)\right\}. \tag{29}$$

If $\underline{Q}$ is set to zero, corresponding to the $\eta \to 0$ limit, the solution of the eigenvalue problem yields the familiar normal modes for disturbances in the wake of an isolated cylinder. Expanding in powers of $\hat{\epsilon}$ around the mode $(\underline{v}^{(0)}, p^{(0)})$ we write

$$(\underline{v}, p) = (\underline{v}^{(0)}, p^{(0)}) + \hat{\epsilon}(\underline{v}', p') + O(\hat{\epsilon}^2), \tag{30}$$

$$\omega = \omega_0 + \hat{\epsilon}\omega' + O(\hat{\epsilon}^2), \tag{31}$$

with $\underline{v}'$ being a complex vector field such that $[\underline{v}', \underline{v}^{(0)^*}] = 0$, and ω' being a complex constant. At order $\hat{\epsilon}$ we have

$$i\omega_0\,\hat{\epsilon}\underline{v}' + L\left(\underline{V}^{(0)}; R\right)\hat{\epsilon}\underline{v}' + \nabla\hat{\epsilon}p' = -i\hat{\epsilon}\omega'\underline{v}^{(0)} + \underline{Q}(\underline{v}^{(0)}), \tag{32}$$

$$\nabla.\hat{\epsilon}\underline{v}' = 0. \tag{33}$$

Following Friedrichs[13], we apply the Fredholm alternative to obtain the unknown shift $\hat{\epsilon}\omega'$, by projecting the equations with the adjoint velocity field $\underline{v}^{(0)^*}$. Writing $\Delta\omega_0$ in place of $\hat{\epsilon}\omega'$ we now have

$$\Delta\omega_0 = -i\left[\underline{Q}(\underline{v}^{(0)}), \underline{v}^{(0)^*}\right] = \Delta\omega + \Delta\Omega, \tag{34}$$

where

$$\Delta\omega = -i\left\{\underline{f}(\underline{v}^{(0)}).\underline{v}^{(0)^*}\right\}_{\underline{r}=\underline{r}_0} = i\epsilon(R_s)\underline{v}^{(0)}(\underline{r}_0).\underline{v}^{(0)^*}(\underline{r}_0), \tag{35}$$

and after integration by parts

$$\Delta\Omega = -i\int_D \hat{\epsilon}V_i' v_j^{(0)}\left\{\frac{\partial v_i^{(0)^*}}{\partial x_j} + \frac{\partial v_j^{(0)^*}}{\partial x_i}\right\} dS. \tag{36}$$

While $\Delta\omega$ may be evaluated explicitly at this stage, the factor $\Delta\Omega$ is a volume integral involving the as-yet unknown steady field $\hat{\epsilon}\underline{V}'$.

In order to determine $\Delta\Omega$ we now consider the inhomogeneous adjoint problem

$$\left(L^*(\underline{V}^{(0)}; R)\underline{V}^{(0)^*}\right)_i - \frac{\partial P^{(0)^*}}{\partial x_i} = v_j^{(0)}\left\{\frac{\partial v_j^{(0)^*}}{\partial x_i} + \frac{\partial v_i^{(0)^*}}{\partial x_j}\right\}, \tag{37}$$

$$\frac{\partial V_i^{(0)^*}}{\partial x_i} = 0, \tag{38}$$

subject to the conditions $\underline{V}^{(0)^*} = \underline{0}$ at the large cylinder walls and at the outer computational boundary. The significance of the solution to this problem can be seen by substituting the left hand side of (37) into the integrand of (36), and then making use of the Lagrange identity (6), whereupon

$$\begin{aligned}\Delta\Omega &= -i\int_D \hat{\epsilon}\underline{V}'.\left\{L^*(\underline{V}^{(0)}; R)\underline{V}^{(0)^*} - \nabla P^{(0)^*}\right\} dS \\ &= -i\int_D \left\{L\left(\underline{V}^{(0)}; R\right)\hat{\epsilon}\underline{V}' + \nabla\hat{\epsilon}P'\right\}.\underline{V}^{(0)^*} dS \\ &\qquad + i\int_D \nabla.\underline{M}(\hat{\epsilon}\underline{V}', \underline{V}^{(0)^*}) dS \\ &= -i\left\{\underline{F}(\underline{V}^{(0)}).\underline{V}^{(0)^*}\right\}_{\underline{r}=\underline{r}_0} = i\epsilon(R_s)\underline{V}(\underline{r}_0).\underline{V}^{(0)^*}(\underline{r}_0).\end{aligned} \tag{39}$$

The first order shift in the complex natural frequency of the vortex shedding motion is thus expressed rather simply as

$$\Delta\omega_0 = i\epsilon(R_s)\left\{\underline{v}^{(0)}.\underline{v}^{(0)^*} + \underline{V}^{(0)}.\underline{V}^{(0)^*}\right\}_{\underline{r}=\underline{r}_0}. \tag{40}$$

The shift in the growth rate is given by $-Im(\Delta\omega_0)$; a negative shift indicates that restabilization is encouraged, while a positive shift indicates that instability is enhanced. Similarly $Re(\Delta\omega_0)/\pi$ indicates the shift in the Strouhal frequency which results from the introduction of the small cylinder. Clearly this shows that as far as global stability is concerned $\underline{v}^{(0)^*}$ weights the influence of the unsteady reaction $-\epsilon\underline{v}^{(0)}$ to the oscillation $\underline{v}^{(0)}$, while $\underline{V}^{(0)^*}$ weights the influence of the steady reaction $-\epsilon\underline{V}^{(0)}$ to the steady flow $\underline{V}^{(0)}$.

After calculating a growth rate $-Im(\Delta\omega_0)$ at the critical condition $R_c = 50$ using (40), it is possible to estimate the shift in the critical Reynolds of the system on the basis of (4) as

$$\Delta R_c = R_c \frac{\frac{Im(\Delta\omega_0)}{0.1}}{1 - \frac{Im(\Delta\omega_0)}{0.1}}. \tag{41}$$

In the upper half of Figure 3 is plotted the change in critical Reynolds number ΔR_c for the flow as a function of the location of a small cylinder 1/10 the diameter of the large one within the flow, as determined by SS from experiment. With $\eta = 1/10$, the lower half of Figure 3 shows the equivalent results calculated with the current theory, based on the solution at a Reynolds number of 50. The forms are clearly close, both qualitatively and quantitatively, especially for the higher critical Reynolds number shift. As noted by SS the stabilizing effect is strongest when the small cylinder is placed just above the vortex sheets emanating from the large cylinder. This rearrangement of the vorticity in the flow has a strong stabilizing effect. This is not the only effect though as the unsteady small cylinder reaction forces, as reflected by the term $\underline{v}^{(0)}.\underline{v}^{(0)^*}$, also play a significant role in the stabilization process.

The most notable differences from experimental results of SS are in the regions of stabilization for Reynolds numbers of only 2 beyond the critical. Experimental results show regions which extend farther from the wake centreline and do not extend upstream of the cylinder as the calculation suggests. The reason for the failure to predict the lateral extent is not clear, although it must be remembered that the approximations made include the assumption of linearity in the flow correction, and a simple point model for the cylinder. The extension of the zone of restabilization well forward of the main cylinder may be explained in terms of vorticity distributions as follows. The presence of the small cylinder acts as a vorticity source $\nabla \times \underline{F}\delta(\underline{r} - \underline{r}_0)$. If the small cylinder is upstream of the large then the wake of this 'vortex dipole' disrupts the vorticity distribution around the large cylinder and as a consequence alters the global stability by some small amount. In the experiment it is possible that noise disrupts the spatial structure of the vortex dipole wake leaving a mixed wake with a weaker effect. The effect is felt however in the noiseless numerical calculation. A small airfoil, rather than a small cylinder, is likely to behave more as a monopole source, and as a result we may speculate that it will have an effect upon stability when positioned upstream. The existence of the region lying on the wake centreline, between 1/2 and 5/2 diameters behind the rear stagnation point is predicted well by the calculation.

Regions where placement of the small cylinder promotes the global instability were not found in the experiment. In the current calculation such regions exist very close to the large cylinder, in front of the separation point and behind the position of maximum surface vorticity. Here our simple model for the force on the small cylinder may be expected to be inaccurate due to the proximity of the large cylinder wall. In view of the experimental evidence it is hard to conclude that a destabilizing effect is possible, though the calculation does suggest a weak effect may be induced.

According to the calculation the tendency of the small cylinder is to reduce the shedding frequency. While this is in qualitative agreement with experiment, the measured reductions at a Reynolds number of 80 may be as large as 30% with the small cylinder ($\eta = 1/10$) located 1.2 cylinder diameters downstream and 1.0 diameter off the centreline. At this point the predicted frequency reduction is only a few percent. Lack of success in capturing the frequency dependence to greater accuracy with the original stability analysis is almost certainly to blame for this discrepancy.

The linear analysis as shown in Figure 3 is clearly quite successful in capturing the effect upon global stability of placing the small cylinder in the wake of the large. Factors which have been ignored include the inertia effect arising from $\hat{\epsilon}\underline{V}'.\nabla\hat{\epsilon}\underline{V}'$ in equation (26), and the fact that the model does not guarantee the no-slip condition on the small cylinder wall. Ideally we would wish to specify a momentum source distribution within the flow which exactly arrests the flow upon that surface within the flow where the small cylinder wall is located. The use of a more sophisticated model for the small cylinder may assist in improving the accuracy of the results though the effort involved may be considerable.

Other control strategies

The appeal of this method lies in the fact that simply solving a pair of adjoint problems, a homogeneous eigenvalue problem and an inhomogeneous problem, provides a map of the influence which unsteady and steady feedback forcing have upon the global stability of the fluid system. Our prescription for the forces $\underline{F}(\underline{V}^{(0)})$ and $\underline{f}(\underline{v}^{(0)})$ used in (35) and (39) was based on our desire to model the effect of the small cylinder.

The effect of a splitter plate laid along the wake centreline can be modelled in a similar way, with the change in global stability simply being the integral over the plate surface of the drag force alignment with the field $\underline{V}^{(0)^*}$. The unsteady loads due to the oscillation $\underline{v}^{(0)}$ could then be weighted by $\underline{v}^{(0)^*}$ to give a further dynamic contribution.

Steady suction on the cylinder surface (in the absense of a second small cylinder) with a velocity distribution $\underline{V}_\phi$ can be shown to induce a change in the eigenvalue ω_0 of

$$i \int_\Gamma \left\{ P^{(0)^*}\underline{n} + \frac{2}{R}\frac{\partial \underline{V}^{(0)^*}}{\partial n} \right\} .\underline{V}_\phi d\Gamma, \qquad (42)$$

where Γ is the cylinder surface and $\underline{n}$ is the local outwards normal. This is derived by reformulating the linear steady flow correction problem (26),(27) by removing the momentum source term, and replacing the boundary condition on the cylinder by $\hat{\epsilon}\underline{V}' = \underline{V}_\phi$. When solving for $\Delta\Omega$ in equation (39) the integral over the flow domain of the divergence of the bilinear concomitant gives the boundary integral (42) over the cylinder surface. The surface velocity distribution is weighted simply by the adjoint normal surface stress associated with the field $(\underline{V}^{(0)^*}, P^{(0)^*})$ which is again the solution of the inhomogeneous adjoint problem (37),(38), with 'no-slip' boundary conditions.

The real part of the adjoint surface stress is a vector field defined over the cylinder surface. It may be regarded as a 'receptivity map' which shows how receptive the instability is to control by suction (or blowing) at any position on the cylinder surface.

Conclusions

A linear stability and perturbation analysis has been performed for the low-Reynolds number flow past a circular cylinder . The restabilizing effect of placing a second smaller cylinder at *any* location within the wake of the first has been quantified by a shift in the critical Reynolds number of the system. The shift predicted from the current theory matches well with experiment[1]. The shedding frequency, when shedding resumes above the new critical Reynolds number, is predicted to be reduced, though the prediction is somewhat below the experimentally-measured results.

Central to the analysis is the formulation and solution of a pair of adjoint problems. The first is simply an adjoint eigenvalue problem whose solutions characterise the the influence of adaptive controllers upon the global stability of the various modes of the system. The solution to the inhomogeneous adjoint problem (37),(38), on the other hand characterises the result of applying a steady control.

It has been suggested how the presence of a splitter plate, or steady wall suction, might also be modelled using the current approach.

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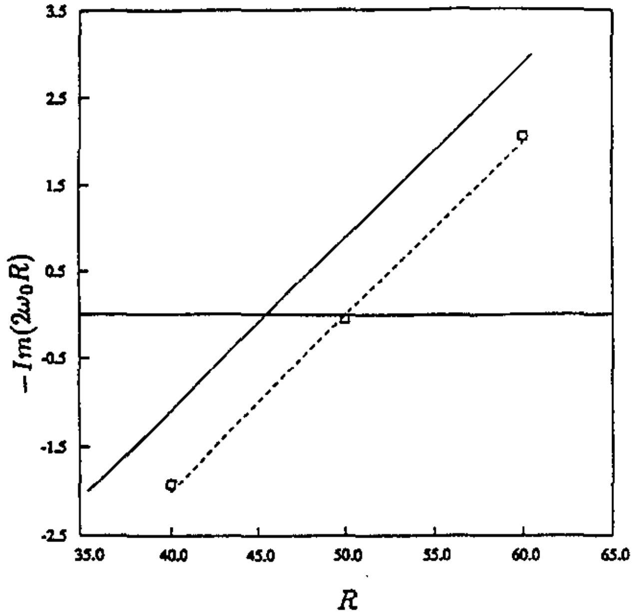

Figure 1. Dimensionless growth rate $-Im(2\omega_0 R) = 0.2(R - R_c)$ as a function of Reynolds number R. Solid line gives fit to experimental results of SS, with $R_c = 46$. Symbols show computed values, with the dashed line showing $0.2(R - R_c)$, with $R_c = 50$.

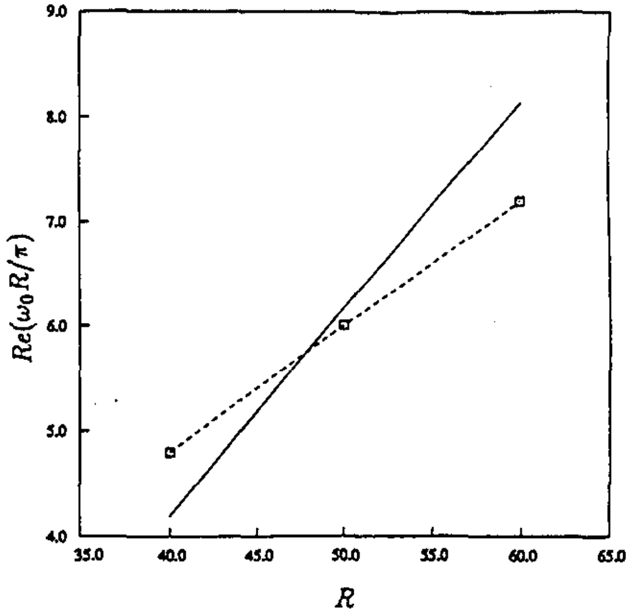

Figure 2. Dimensionless frequency $Re(\omega_0 R/\pi)$ as a function of Reynolds number R. Solid line gives a fit to the experimental results of SS (see also Williamson[9]). Symbols show computed values.

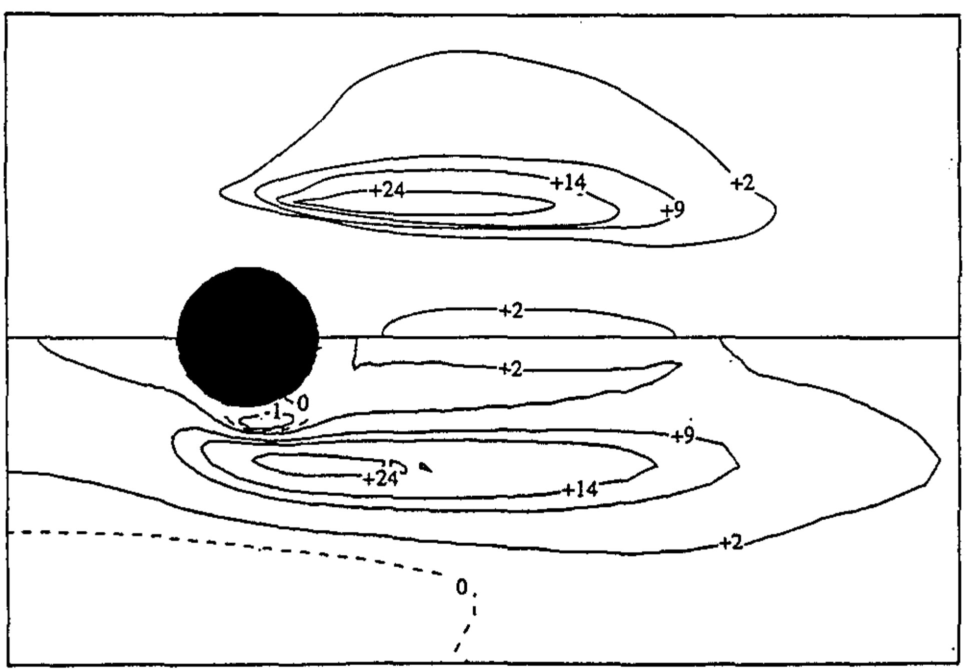

Figure 3. Shift in the critical Reynolds number ΔR_c for the onset of vortex shedding, as a result of placement of a small cylinder 1/10 the diameter of the large in the wake. Upper half plane shows experimental results of SS. Lower half plane shows computed results.